REGGEIZED RESONANCE MODEL FOR THE PRODUCTION AMPLITUDE

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We present a generalization of Veneziano's model to the five particle (production) amplitude. The amplitude consists of multiple resonance exchanges and shows multi - Regge behaviour in all channels. The residues at the poles have the correct polynomial dependence on the momentum transfer variables. The model also has partial crossing symmetry. Some conclusions and possible applications are pointed out.

The five-point amplitude is schematically represented in fig. 1, where the momentum entering the i'th line is denoted by P_i. The five independent scalars are chosen to be

$$s_{i,i+1} \equiv (p_i + p_{i+1})^2, \qquad i = 1,2,\ldots,5, \quad p_i^2 = 1, \quad p_6 \equiv p_1 \tag{1}$$

and the amplitude is denoted by $F(s_{12}, s_{23}, s_{34}, s_{45}, s_{51})$.

For simplicity, the particles are taken to be identical and of spin zero and the Regge trajectories on which the resonances lie are assumed to be identical in all channels.

Just as in the Veneziano model [1], the trajectory $l = \alpha(s)$ must be a linear increasing function of s. We now list the requirements we want the model to satisfy:

a) F has simple poles in the five variables $s_{i,i+1}$, which correspond to resonances in a given channel or to the crossed poles giving rise to the Regge behaviour in the same channel. Two variables s_{ij} and s_{nm} can simultaneously develop a pole if there exists a Feynman diagram of the form given in fig. 2. This is only possible if i,j,m,n, are all different. This is graphically represented by fig. 3, where the vertices of the pentagon label the indices of the sub-energies $s_{i,i+1}$, and the dotted diagonals correspond to the allowed double poles;

b) for definiteness, consider the double pole in the variables s_{12} and s_{45} corresponding to fig. 2. Let the spin of the leading pole in the 12 channel be j and in the 45 channel j', with masses $\alpha^{-1}(j)$ and $\alpha^{-1}(j')$ respectively. The amplitude near $s_{12} = \alpha^{-1}(j)$ and $s_{45} = \alpha^{-1}(j')$ must have the form

$$F \sim \frac{1}{(s_{12} - \alpha^{-1}(j))(s_{45} - \alpha^{-1}(j'))} \sum_{K=0}^{K=\min(j,j')} c_k (s_{15})^K (s_{23})^{j-K} (s_{34})^{j'-K} \tag{2}$$

where c's are constants. This condition, easily derivable from Feynman rules, guarantees that there are no "ancestors" to the leading Regge trajectory. Of course, it allows parallel daughters. The same condition also applies to other pairs of variables by a cyclic interchange of variables or by rotating the pentagon of fig. 3;

c) the model must reggeize both in the single and the double Regge limits in all channels. That means, choosing the particular graphs of fig. 4,

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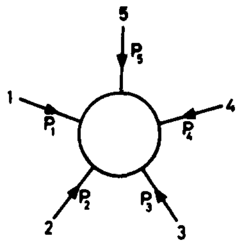

Fig.1.

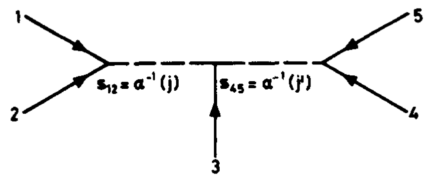

Fig. 2.

$$F \to \{1 + \exp(-i\pi\alpha(s_{15}))\}(s_{45})^{\alpha(s_{15})} f(h, s_{23}, s_{34}, s_{15}), \quad s_{12} \to \infty, \quad s_{45} \to \infty, \quad h = s_{45}/s_{12} = \text{fixed} \tag{3a}$$

$$F \to \{1 + \exp(-i\pi\alpha(s_{23}))\}\{1 + \exp(-i\pi\alpha(s_{15}))\}(s_{34})^{\alpha(s_{23})}(s_{45})^{\alpha(s_{15})} g(K, s_{23}, s_{15}),$$
$$s_{34} \to \infty, \quad s_{45} \to \infty, \quad \frac{s_{34}\, s_{45}}{s_{12}} = K = \text{fixed}, \; s_{15}, s_{23} \text{ fixed} \tag{3b}$$

where K is simply related to the Toller variable [2]. These relations must also the true under any cyclic permutation.

The formula we propose is an extension of the integral representation for the Euler function which appears in the Veneziano model,

$$B(-\alpha(s), -\alpha(t)) = \int_0^1 du\, u^{-\alpha(s)-1}(1-u)^{-\alpha(t)-1} \; . \tag{4}$$

We generalize the above formula to the following:

$$F = \int_0^1 \int_0^1 \frac{du_i du_j}{1-u_i u_j}\, u_1^{-\alpha_{12}-1}\; u_2^{-\alpha_{23}-1}\; u_3^{-\alpha_{34}-1}\; u_4^{-\alpha_{45}-1}\; u_5^{-\alpha_{51}-1} \tag{5a}$$

where $\alpha_{12} \equiv \alpha(s_{12}) = as_{12} + b$ etc., and indices i and j are any two non-successive integers, counting 6 and 1 equivalent. In the following we use the same F for the functions of s_{ij} and α_{ij}. u_i and u_j correspond to the dotted diagonal lines in the pentagon of fig. 3. The variables u_i satisfy the following constraints:

$$u_i = 1 - u_{i-1} u_{i+1}, \quad i = 1, \ldots, 5, \quad u_6 \equiv u_i \, . \tag{5b}$$

Although these are five equations, only three are linearly independent and hence all the u's can be ex-

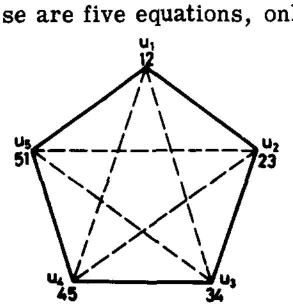

Fig. 3.

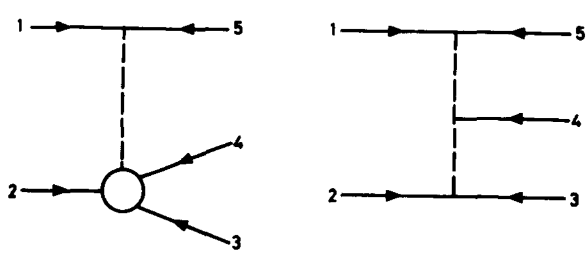

Fig. 4.

pressed in terms of two of them. By a change of variable, it is easily seen that the right-hand side of (5a) is independent of the choice of i and j, and that it is symmetric under the cyclic interchange of the indices from 1 to 5. Therefore, it is sufficient to prove properties b) and c) in one channel only; because of the cyclic symmetry, the proof is the same in other channels. Having established the symmetry of formula (5a), we now write it in an unsymmetrical looking form, after elimination of some variables by (5b),

$$F=\int_0^1 du_1 \int_0^1 du_4 \; u_1^{-\alpha_{12}-1} \; u_4^{-\alpha_{45}-1} \left(\frac{1-u_1}{1-u_1u_4}\right)^{-\alpha_{23}-1} \left(\frac{1-u_4}{1-u_1u_4}\right)^{-\alpha_{34}-1} (1-u_1u_4)^{-\alpha_{15}-2} . \qquad (6)$$

This formula is convergent when all the exponents have a negative real part and, therefore, it is analytic in all the variables in this region. When $\mathrm{Re}(\alpha_{12})$ and $\mathrm{Re}(\alpha_{45})$ become positive, the integral diverges at the lower limits of integration, and gives rise to poles in the variables s_{12} and s_{45}. From (5b) it is clear that u_i and u_{i+1} cannot vanish simultaneously, so that s_{ij} and s_{kl} cannot simultaneously develop poles if they share an index. On the other hand, any two non-adjacent variables like u_1 and u_4 can approach the lower limits of integration simultaneously, so that s_{12} and s_{45} can develop poles at the same time. This verifies condition a).

To study the simultaneous poles in variables s_{12} and s_{45}, we write

$$F=\left(\int_0^\epsilon\int_0^\epsilon + \int_\epsilon^1\int_0^\epsilon + \int_0^\epsilon\int_\epsilon^1 + \int_\epsilon^1\int_\epsilon^1\right) du_1\, du_4 \,\{\text{integrand}\} , \quad 0<\epsilon<1 . \qquad (7)$$

Only the first term leads to poles in both s_{12} and s_{45}; the other terms are either analytic in s_{12}, s_{45} or both. We now expand the last three factors in the integrand in a double Taylor series, keeping α_{45} and α_{12} negative, integrate term by term and obtain:

$$F=\sum_{k,l,m=0}^{\infty} \frac{(\epsilon)^{k+m-\alpha_{12}}}{k+m-\alpha_{12}} \frac{(\epsilon)^{l+m-\alpha_{45}}}{l+m-\alpha_{45}} \times (-\alpha_{23}-1)\ldots(-\alpha_{23}-k) \times (-\alpha_{34}-1)\ldots(-\alpha_{34}-l) \times$$
$$\times (\alpha_{23}+\alpha_{34}-\alpha_{15})\ldots(\alpha_{23}+\alpha_{34}-\alpha_{15}-m+1) + \text{terms without multiple poles in } s_{12} \text{ and } s_{45} . \qquad (8)$$

This formula, continued to positive values of α_{45} and α_{12}, displays the poles at non-negative integers in these variables.

To check condition b), we identify $j=k+m$, $j'=l+m$ and observe that the leading term of the polynomial that multiplies this pole is of the form $(s_{15})^a (s_{23})^{j-a} (s_{34})^{j'-a}$, and hence condition b) holds.

An alternative and perhaps better way of deriving the above results proceeds through functional recursion relations. They can easily be derived by partial integrations with respect to u_1 and u_4 and also by the rearrangement of the variables in the integral:

$$F(\alpha_{12},\alpha_{23},\alpha_{34},\alpha_{45},\alpha_{51}) = F(\alpha_{12},\alpha_{23}+1,\alpha_{34},\alpha_{45},\alpha_{51}+1) - F(\alpha_{12}-1,\alpha_{23}+1,\alpha_{34},\alpha_{45},\alpha_{51}+1) ,$$

$$F(\alpha_{12},\alpha_{23},\alpha_{34},\alpha_{45},\alpha_{51}) = F(\alpha_{12},\alpha_{23},\alpha_{34},\alpha_{45},\alpha_{51}+1) - F(\alpha_{12}-1,\alpha_{23},\alpha_{34},\alpha_{45}-1,\alpha_{51}+1) ,$$

$$F(\alpha_{12},\alpha_{23},\alpha_{34},\alpha_{45},\alpha_{51}) = \frac{1}{\alpha_{12}} \{(\alpha_{23}+1)F(\alpha_{12}-1,\alpha_{23}+1,\alpha_{34}-1,\alpha_{45},\alpha_{51}+1) +$$
$$+ (\alpha_{15}-\alpha_{34}+1)F(\alpha_{12}-1,\alpha_{23},\alpha_{34},\alpha_{45}-1,\alpha_{51}+1)\} ,$$

$$F(\alpha_{12},\alpha_{23},\alpha_{34},\alpha_{45},\alpha_{51}) = \frac{\alpha_{51}-\alpha_{23}+1}{\alpha_{45}} F(\alpha_{12}-1,\alpha_{23},\alpha_{34},\alpha_{45}-1,\alpha_{51}+1) + \frac{\alpha_{34}+1}{\alpha_{45}} F(\alpha_{12},\alpha_{23}-1,\alpha_{34}+1,\alpha_{45}-1,\alpha_{51}+1) . \tag{9}$$

Other functional equations can be obtained by all cyclic permutation of the arguments in the above equations. By repeated use of these equations, one can continue F from its original domain of analyticity $\mathrm{Re}\,\alpha_{ij} < 0$ to the whole complex plane. One then encounters simple poles coming from the denominators in the last two formulas, and these clearly have polynomial residues. Conditon b) can be established using induction on (9).

As for reggeization, note that (6) does not have the signature factors, since there is no crossing symmetry between say lines 1 and 5 and 2 and 3 in fig. 4. In any given channel, this can easily be fixed by making the model crossing symmetric between the appropriate lines "by hand". For example, for the channel chosen in fig. 4, we define a new function:

$$\begin{aligned}\tilde{F}(s_{12},s_{23},s_{34},s_{45},s_{51}) &= F(s_{12},s_{23},s_{34},s_{45},s_{51}) + F(s_{45}-s_{32}-s_{12}+3, s_{23}, s_{15}-s_{23}-s_{34}+3, s_{45}, s_{51}) + \\ &+ F(s_{34}-s_{12}-s_{15}+3, s_{23}, s_{34}, s_{23}-s_{15}-s_{45}+3, s_{51}) + \\ &+ F(s_{12}-s_{34}-s_{45}+3, s_{23}, s_{15}-s_{23}-s_{34}+3, s_{23}-s_{15}-s_{45}+3, s_{51})\end{aligned} \tag{10}$$

where $s_{45}-s_{32}-s_{12}+3 = s_{13}$, $s_{15}-s_{23}-s_{34}+3 = s_{24}$, etc. If F reggeizes so does $\tilde{F}$, and with correct signature factors. The model can of course be made completely crossing symmetric, but then the proof of reggeization is much more difficult and will not be attempted here.

We now give the sketch of a proof of double reggeization, the same argument can also be used to establish single reggeization. In eq. (6) take all α's real and negative, and make the following change of variables:

$$u_1 = \exp\left\{\frac{-xy}{\alpha_{34}\alpha_{45}}\right\}, \quad u_4 = \exp\left\{\frac{y}{\alpha_{45}}\right\}$$

which gives

$$F = (-\alpha_{34})^{\alpha_{23}} (-\alpha_{45})^{\alpha_{15}} I$$

where

$$I = \int_0^\infty dx \int_0^\infty dy \exp\{-y + \frac{xy}{\tilde{k}}\}(x)^{-\alpha_{23}-1}(y)^{-\alpha_{15}-1} \times [K(x',y')]^{-\alpha_{23}-1} L(x',y',\alpha_{34})[M(x',y')]^{-\alpha_{15}-2} \tag{11}$$

with

$$x' = -x/\alpha_{34}, \quad y' = -y/\alpha_{45}, \quad \tilde{k} = \alpha_{34}\alpha_{45}/\alpha_{12} = \text{fixed} < 0$$

$$K(x',y') = \frac{1-\exp\{-x'y'\}}{1-\exp\{-x'y'-y'\}} \frac{1}{x'} , \quad M(x',y') = \frac{1-\exp\{-x'y'-y'\}}{y'} ,$$

$$L(x',y',\alpha_{34}) = \left\{\frac{1-\exp\{-y'\}}{1-\exp\{-x'y'-y'\}}\right\}^{-\alpha_{34}-1}$$

The range of both x and y and y' and y' is between 0 and $+\infty$. In this range, the following statements can explicitly be verified:

$$|K| < C , \quad |L| < C , \quad |M| < C$$

where C is a fixed constant independent of all variables $(0 \leqslant x' \leqslant \infty,\ 0 \leqslant y' \leqslant \infty,\ \alpha_{34} < 0)$

$$\lim_{x',y' \to 0} K(n',y') = 1\ , \qquad \lim_{x',y' \to 0} M(x',y') = 1 \tag{12}$$

$$\lim_{\substack{x' \to 0, y' \to 0 \\ x,y = \text{fixed}, \alpha_{34} \to -\infty}} L(x',y',\alpha_{34}) = \exp\{-x\}\ .$$

The above limits are all uniform in both variables in the neighbourhood of $x',y' = 0$. We can now write

$$I = \left\{\int_0^P \int_0^P + \int_P^\infty \int_0^P + \int_0^P \int_P^\infty + \int_P^\infty \int_P^\infty \right\} \mathrm{d}x\mathrm{d}y \times \{\text{integrand}\} = I_0(P) + I_1(P) + I_2(P) + I_3(P) \tag{13}$$

where P is a constant independent of all s's. Now, given an ϵ, we choose P so large that

$$|I_1| < \epsilon \quad |I_2| < \epsilon \quad |I_3| < \epsilon\,, \text{ independent of } s_{34} \text{ and } s_{45} \tag{14}$$

which follow from the fact that L, M, N are bounded.

At the same time, keeping P fixed and taking s_{34} and s_{45} sufficiently large, we can satisfy the following:

$$\left|I_0(P) - \int_0^P \int_0^P \mathrm{d}x\mathrm{d}y \exp\{-x-y+\frac{xy}{\tilde{k}}\}(x)^{-\alpha_{23}-1}(y)^{-\alpha_{15}-1}\right| < \epsilon \tag{15}$$

which follows from the existence of uniform limits for K, L, M. Finally we can clearly have

$$\left|\left\{\int_0^\infty \int_0^\infty - \int_0^P \int_0^P\right\} \mathrm{d}x\,\mathrm{d}y \exp\{-x-y+\frac{xy}{\tilde{k}}\}(x)^{-\alpha_{23}-1}(y)^{-\alpha_{15}-1}\right| < \epsilon\ . \tag{16}$$

Combining eqs. (13), (14), (15) and (16), the following limit is derived

$$\lim_{\substack{s_{34} \to -\infty \\ s_{45} \to -\infty \\ \tilde{k} \text{ fixed}}} I = g \tag{17}$$

which proves reggeization for the first term in (10) and also gives

$$g(K, s_{23}, s_{15}) = (a)^{\alpha_{23}+\alpha_{15}} \int_0^\infty \int_0^\infty (x)^{-\alpha_{23}-1}(y)^{-\alpha_{15}-1} \exp\{-x-y+\frac{xy}{aK}\}\,\mathrm{d}x\mathrm{d}y \tag{18}$$

where a is the slope of the trajectory and g is the residue function defined by (3b).

The above argument goes through even when α_{34} and α_{45} are complex as long as $\mathrm{Re}\,\alpha_{34} < 0$ and $\mathrm{Re}\,\alpha_{45} < 0$. To reach the right half plane in these variables, one has to rotate the line of integration of x and y in eq. (11) from the positive axis to a complex direction. In this manner, one can establish (17) for any complex direction with the exception of positive real axis. As a check, one may introduce the expected Regge behaviour (3b) into (9). One then gets functional equations for g and verifies that they are the same as those derived from (18).

The model we have presented satisfies the duality conditions of Dolen et al. [3] for the five-point production amplitude. The amplitude consists of only multiple resonance exchanges and yet it reggeizes. As a result, it resolves the paradox of Deck effect versus direct resonance model in the manner conjectured by Chew and Pignotti [4]. Finally, it predicts the form of the residue function in case of multiple reggeization. The dependence on the variable K given by eq. (18) is an interesting consequence.

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